Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 14

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Textbook Sections: 7.7

Today's Topic: Estimating Integrals

A general issue: Frequently, we just can't compute integrals explicitly. This can happen when, for example,

1) we can't find an explicit anti-derivative
2) The function we integrate is only partially known – eg we can only sample it in experiments a finitely many places.

Nevertheless: we still want to understand these integrals numerically. That is, we still hope to approximate their values.

Back to Basics:

To start off, recall that when $f:[a,b]\to\mathbb{R}$ is cts, we defined

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)\Delta x_i$$

where the limit is taken over Riemann sums for $f(x)$ over $[a,b]$ whose partition intervals $\{[x_{i-1},x_i]\}_{i=1}^N$ have lengths $\Delta x_i \longrightarrow 0$ as $N\to\infty$.

Therefore: it is obvious that for N large enough, a Riemann sum $\sum_{i=1}^N f(x_i^*)\Delta x_i$ approximates $\int_a^b f(x)\,dx$.

Let's touch on a couple examples of this, and think about how to do it more effectively.

In all these setups, lets let $[a,b] = [x_0,x_1] \cup [x_1,x_2] \cup \cdots \cup [x_{N-1},x_N]$ so that $x_0 = a$, $x_N = b$, and $\Delta x_i = x_i - x_{i-1} = \frac{b-a}{N}$.

(I) If we choose $x_i^* = x_{i-1}$ (the left endpts), we call

$$L_N = \sum_{i=1}^{N} f(x_i^*)\Delta x_i = \sum_{i=1}^{N} f(x_{i-1})\left(\frac{b-a}{N}\right)$$

(II) If we choose $x_i^* = x_i$ (the right endpts), we call

$$R_N = \sum_{i=1}^{N} f(x_i^*)\Delta x_i = \sum_{i=1}^{N} f(x_i)\left(\frac{b-a}{N}\right)$$

(III) If we choose the midpoint $x_i^* = \frac{x_i + x_{i-1}}{2}$, we call

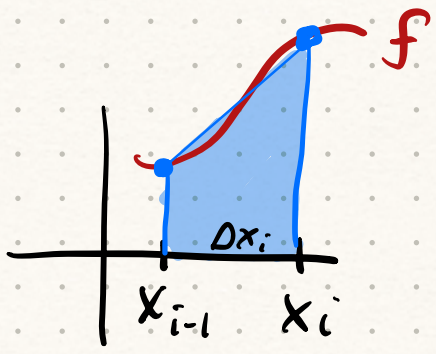
$$M_N = \sum_{i=1}^{N} f\left(\frac{x_i + x_{i-1}}{2}\right)\left(\frac{b-a}{N}\right)$$

(IV) If we average L_N and R_N, we get the trapezoidal approx:

$$T_N := \frac{L_N + R_N}{2} = \sum_{i=1}^{N} \frac{f(x_{i-1}) + f(x_i)}{2} \cdot \left(\frac{b-a}{N}\right)$$

the name comes from the fact that

$$\frac{f(x_{i-1}) + f(x_i)}{2} \cdot \left(\frac{b-a}{N}\right) = \text{Area of these trapezoids:}$$



General Fact: When $N \to \infty$,

$$L_N, R_N, M_N, T_N \longrightarrow \int_a^b f(x)\,dx,$$

but they are not all as good at approximating the integral.

Error Bounds: Suppose $K \in \mathbb{R}$ is big enough so that $|f''(x)| \leq K$ for every $x \in [a,b]$ (this bound controls how "tightly" the graph of $f(x)$ can bend). Set

$$E(M_N) = \left| \int_a^b f(x)\,dx - M_N \right|$$

$$E(T_N) = \left| \int_a^b f(x)\,dx - T_N \right|$$

the errors

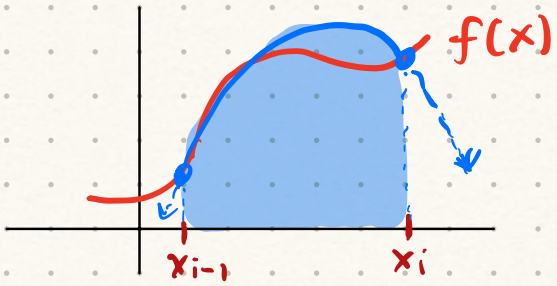
Then

- $E(T_N) \leq \dfrac{K(b-a)^3}{12N^2}$
- $E(M_N) \leq \dfrac{K(b-a)^3}{24N^2}$

A more "exotic" approach: Simpson's Rule

In essence, L_N, R_N, M_N, and T_N all approximate $\int_a^b f(x)\,dx$ by approximating the area under f on each partition segment $[x_{i-1}, x_i]$ by the area of a rectangle or trapezoid (which themselves have easily computable areas). However, most of the time f won't be linear, so these basic approximating shapes are limited in terms of what they can efficiently approximate.

Simpson's Rule (see the HW for details) gives an alternative approach, by using parabolas to approximate the graph of $f(x)$ over each subinterval.



Fact: If $|f^{(4)}(x)| \leq K$ on $[a,b]$, then

$$|E_S| \leq \frac{K(b-a)^5}{180\,n^4}$$